Thick subcategories for classical Lie superalgebras

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Motivating Example

$G =$ finite group, $\quad k =$ field of characteristic $p > 0$

mod(kG), the category of finitely-generated kG-modules, is usually "wild" — the indecomposable representations cannot be classified.

Q: Can one make less-refined classifications that are still useful?

There are only finitely many projective indecomposables, and these are understood. Suggests working "modulo projectives":

stmod(kG) $=$ the stable module category of f.g. kG-modules: same objects, but put equivalence relation on morphisms so that proj $\cong$ 0. This is a triangulated category.

A thick subcategory of stmod(kG) is a full triangulated subcategory closed under taking direct summands.

A tensor ideal is a thick subcategory closed under tensoring by arbitrary objects in stmod(kG).

Q: Can one classify tensor ideals in stmod(kG)?

A: Yes — Benson–Carlson–Rickard, 1997. A key ingredient is geometry, specificially, support varieties.

$R := \mathrm{H}^\bullet(G, k)$, a ring under Yoneda product

$\mathrm{Ext}^\bullet_{kG}(M, M)$ is a f.g. R-module, $M \in \mathrm{mod}(kG)$

$I_M := \mathrm{Ann}_R \, \mathrm{Ext}^\bullet_{kG}(M, M)$

$V_G(M) := \mathrm{MaxSpec}(R/I_M)$, the support variety of M, a subvariety of $V_G := \mathrm{MaxSpec}(R)$

A subset Y of a variety X is specialization-closed if Y is a union of closed subsets.

Theorem (Benson-Carlson-Rickard)

The tensor ideals of stmod(kG) *are in bijection with the homogeneous specialization-closed subsets of* V_G*. Such a subset* Y *corresponds to the full subcategory of modules* M *with* $V_G(M) \subset Y$*.*

The proof involves numerous ingredients besides support varieties, including Rickard's idempotent (aka localizing) functors, which require passing to a category of infinitely-generated modules.

A General Framework

In a series of papers, Benson–Iyengar–Krause have developed a general framework for establishing classification results for thick subcategories, tensor ideals, localizing subcategories (closed under arbitrary coproducts), etc., in a (tensor) triangulated category $\mathbf{T}$.

Their setup involves a graded-commutative ring R "acting" on $\mathbf{T}$. In examples, R is typically a cohomology ring acting in a compatible way on the graded ring of self-extensions for each object in the category.

They introduce support varieties $\operatorname{supp}_R(M) \subset \operatorname{Spec} R$, for $M \in \mathbf{T}$.

They then give conditions under which support classifies interesting kinds of subcategories in terms of subsets of $\operatorname{Spec} R$.

Another Example: $\mathfrak{gl}(1|1)$

$\mathfrak{g} = \mathfrak{gl}(1|1) =$ Lie superalgebra of 2×2 complex matrices:

$$\mathfrak{g}_{\bar{0}} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \ \mathfrak{g}_{\bar{1}} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \ \mathfrak{g}_{-1} = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, \ \mathfrak{g}_{1} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

$T =$ invertible 2×2 diagonal matrices

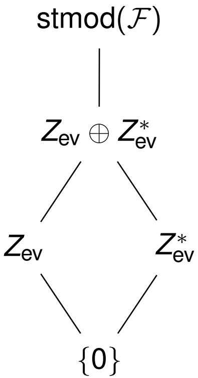
$\mathcal{F} = \mathcal{F}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}) =$ f.d. $\mathfrak{g}$-modules completely reducible over $\mathfrak{g}_{\bar{0}}$

The indecomposable modules in $\mathcal{F}$ have been classified, and the decomposition of their tensor products was worked out by Götz–Quella–Schomerus in 2005.

Four types of indecomposables: irreducibles, projective covers, “zig-zag modules”, and “dual zig-zag modules”. The zig-zag modules (and their duals) further separate into even and odd length cases.

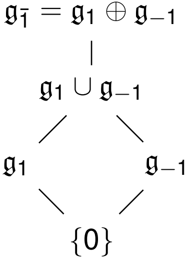
Using Götz *et. al.*, the tensor ideals in stmod($\mathcal{F}$) are as follows:

$\mathfrak{gl}(1|1)$ Tensor Ideals



Z_{ev} (resp. Z_{ev}^*) = direct sums of even zigzag (resp. even dual zigzag) modules and projectives

On the other hand, we have developed a support theory for $\mathcal{F}$, whose values are subvarieties of $\mathfrak{g}_{\bar{1}}$. The closed, conical, T-invariant subvarieties of $\mathfrak{g}_{\bar{1}}$ are as follows:



Moreover, the obvious bijection between these two pictures is given in the same way as in the finite group setting, with a subvariety Y corresponding to the full subcategory of modules whose support is contained in Y.

For $\mathfrak{gl}(1|1)$, we showed in previous work on "detecting subalgebras" that $\operatorname{Spec} \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C}) \cong \mathbb{A}^1$, and therefore only has two closed conical subsets: $\{0\} \subset \mathbb{A}^1$.

This and other examples show that the spectrum of the cohomology ring may not be big enough to afford a support variety theory which can classify the tensor ideals, in the Lie superalgebra case.

Question

Can the Benson–Iyengar–Krause framework be generalized to include situations where the geometry does not necessarily arise from the action of a ring R on the triangulated category $\mathbf{T}$*?*

A General Setting

$\mathcal{D}$ = symmetric monoidal tensor triangulated category whose objects are "module-like"

$\mathbf{T} = \text{stmod}(\mathcal{D})$

X = Noetherian algebraic variety, such that every irreducible closed subvariety $V \subset X$ has a generic point (i.e. $x \in V$ with $\overline{\{x\}} = V$)

$\mathcal{D} \ni M \mapsto X_M \subset X$ a subvariety satisfying the usual properties of a support theory (direct sum, tensor product, etc.)

$\mathcal{S}$ = all specialization-closed conical subsets of X, possibly satisfying some additional properties

Tensor($\mathcal{M}$) = the tensor ideal in $\mathbf{T}$ generated by $\mathcal{M} \subset \mathcal{D}$

Assume the following:

Realization: For each closed set $V \in \mathcal{S}$ there exists $M \in \mathcal{D}$ with $X_M = V$.

Hopkins Property: For $M \in \mathcal{D}$, $\text{Tensor}(M) = \{N \in \mathcal{D} : X_N \subset X_M\}$.

Then we have the following:

Theorem

There is a pair of mutually inverse maps

$$\{\textit{tensor ideals of } \mathbf{T}\} \underset{\Theta}{\overset{\Gamma}{\rightleftarrows}} \mathcal{S},$$

given by $\Gamma(\mathcal{C}) = \bigcup_{M \in \mathcal{C}} X_M, \qquad \Theta(V) = \{N \in \mathcal{D} : X_N \subset V\}.$

Application to $\mathfrak{gl}(m|n)$

Let $\mathfrak{g} = \mathfrak{gl}(m|n) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \right\} \qquad \mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\} \qquad \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right\}.$$

The group $G_0 \cong GL(m) \times GL(n)$ acts on $\mathfrak{g}_1$ by $(A, B) \cdot x = AxB^{-1}$.

$\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1$, a classical Lie superalgebra

$\mathcal{F} = \mathcal{F}(\mathfrak{p}, \mathfrak{g}_0) =$ f.d. $\mathfrak{p}$-modules completely reducible over $\mathfrak{g}_0$

Given $M \in \mathcal{F}$, define the $\mathfrak{g}_1$-rank variety of M,

$$\mathcal{V}_{\mathfrak{g}_1}(M) = \left\{ x \in \mathfrak{g}_1 : M|_{\langle x \rangle} \text{ is not projective} \right\} \cup \{0\} \subset X := \mathfrak{g}_1.$$

$\mathcal{V}_{\mathfrak{g}_1}(\ \ )$ satisfies the standard properties of a support variety theory.

Theorem

For $\mathfrak{p}$ and $\mathcal{F}$ as above,

$$\{\ \textit{tensor ideals of } \mathrm{stmod}(\mathcal{F})\ \} \longleftrightarrow \{\ \textit{closed } G_0\textit{-invariant subsets of } \mathfrak{g}_1\ \}$$

Realization: The G_0-orbit closures are the determinantal varieties

$$(\mathfrak{g}_1)_k := \{\, x \in \mathfrak{g}_1 : \mathrm{rank}(x) \leq k\,\}, \quad 0 \leq k \leq \min(m, n).$$

Using earlier work of Duflo-Serganova, we showed that $\mathcal{V}_{\mathfrak{g}_1}(L) = (\mathfrak{g}_1)_k$ when L is a finite-dimensional simple module of atypicality k.

Hopkins Property: This follows using idempotent functors, along the same lines as the proof in the finite groups case.