Extensions for Generalized Current Algebras

Brian D. Boe[1] Christopher M. Drupieski[2] Tiago R. Macedo[3]
Daniel K. Nakano[1]

[1]University of Georgia

[2]DePaul University

[3]University of Ottawa and Federal University of São Paulo

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Setup & Background

$\mathfrak{g}$ = simple Lie algebra / $\mathbb{C}$

A = fin. gen. comm. assoc. unital $\mathbb{C}$-alg. (e.g. $\mathbb{C}[t]$)

$\mathfrak{g}[A] = \mathfrak{g} \otimes A$, a **generalized current algebra**; Lie algebra via

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg, \quad x, y \in \mathfrak{g}, \ \ f, g \in A$$

When $A = \mathbb{C}[t]$, denote $\mathfrak{g}[A] =: \mathfrak{g}[t]$, the **current algebra**.

- Gell-Mann, 1960s, electromagnetic currents of strongly interacting particles
- When $A = \mathbb{C}[t]$ or $\mathbb{C}[t, t^{-1}]$, rep'n. theory of $\mathfrak{g}[A]$ well studied, as important subalgebras of affine Lie algebra $\hat{\mathfrak{g}}$
- Lie algebra cohomology of $\mathfrak{g}[A]$ not well understood

Two Problems

Q1: Assume M is a f.d. $\mathfrak{g}[A]$-module. Is $\mathrm{H}^n(\mathfrak{g}[A], M)$ f.d.?

Q2: Describe $\mathrm{Ext}^n_{\mathfrak{g}[A]}(L_1, L_2)$ for simple f.d. $\mathfrak{g}[A]$-modules L_1, L_2.

Note: the simple f.d. $\mathfrak{g}[A]$-modules have been classified by Chari-Fourier-Khandai (2010): they are tensor products of “evaluation modules” obtained from simple f.d. $\mathfrak{g}$-modules.

Basic Notation

$\mathfrak{g} \supset \mathfrak{h} =$ Cartan subalgebra

$W =$ Weyl group, $w_0 =$ longest element, $\theta =$ highest root

$P^+ =$ dominant int. weights, $V(\lambda) =$ simple f.d. $\mathfrak{g}$-module, h.w. $\lambda \in P^+$

$V(\lambda)^* = V(\lambda^*)$, $\lambda^* := -w_0\lambda$

$\mathfrak{g} \cong \mathfrak{g} \otimes \mathbb{C} \cdot 1 \leq \mathfrak{g}[A]$

If $A \triangleright A_+$ augmented, $\mathfrak{g}[A_+] := \mathfrak{g} \otimes A_+$. Write $\mathfrak{g}[t]_+ := \mathfrak{g} \otimes t\,\mathbb{C}[t]$.

Simple Finite-Dimensional Modules

$\forall \mathfrak{m} \in \mathrm{MaxSpec}(A)$, $A \to A/\mathfrak{m} \cong \mathbb{C}$ induces $\mathrm{ev}_{\mathfrak{m}} : \mathfrak{g}[A] \to \mathfrak{g}$ "evaluation."

$\mathfrak{g}$-module $V \rightsquigarrow \mathrm{ev}_{\mathfrak{m}}^* V = \mathfrak{g}[A]$-module; note $\mathrm{ev}_{\mathfrak{m}}^* V(0) = \mathbb{C}\ \ \forall \mathfrak{m}$.

$\mathcal{P} = \{\, \pi : \mathrm{MaxSpec}(A) \to P^+ \mid \pi \text{ finitely supported} \,\}$

$\mathcal{V}(\pi) = \bigotimes_{\mathfrak{m} \in \mathrm{MaxSpec}(A)} \mathrm{ev}_{\mathfrak{m}}^* V(\pi(\mathfrak{m})) =$ a f.d. $\mathfrak{g}[A]$-module

$\mathcal{V}(\pi)^* = \mathcal{V}(\pi^*)$ where $\pi^*(\mathfrak{m}) := \pi(\mathfrak{m})^* = -w_0\pi(\mathfrak{m})$

Theorem (Chari-Fourier-Khandai, 2010)

The $\mathcal{V}(\pi)$, $\pi \in \mathcal{P}$, form a complete irredundant set of finite-dimensional irreducible $\mathfrak{g}[A]$-modules.

Theorem (1)

A augmented, $M = \mathfrak{g}[A]$-module, finitely semisimple for $\mathfrak{g}$:

$$\mathrm{H}^n(\mathfrak{g}[A], M) \cong \bigoplus_{i+j=n} \mathrm{H}^i(\mathfrak{g}[A_+], M)^{\mathfrak{g}} \otimes \mathrm{H}^j(\mathfrak{g}, \mathbb{C}).$$

- For $A = \mathbb{C}[t]$, recovers a result of Fialowski-Malikov (2004) giving degree n extensions between single evaluation modules
- So f.d. of H^n is reduced to that of $\mathrm{H}^i(\mathfrak{g}[A_+], M)^{\mathfrak{g}}$ for $0 \leq i \leq n$
- Proof uses relative Lie algebra cohomology spectral sequences

Next: specialize to $A = \mathbb{C}[t], M = \mathbb{C}$...

Garland-Lepowsky results

W_a = affine Weyl group

W_a^1 = minimal length coset reps for W in W_a

Theorem (Garland-Lepowsky, 1979)

For all $i \geq 0$, as $\mathfrak{g}$-modules:

$$\mathrm{H}^i(\mathfrak{g}[t]_+, \mathbb{C}) \cong \bigoplus_{\substack{w \in W_a^1 \\ \ell(w)=i}} V(\lambda_w)^*,$$

where $\lambda_w \sim w \cdot 0$, and the $V(\lambda_w)^*$ are non-isomorphic $\mathfrak{g}$-modules.

For small i, one can work out the right hand side explicitly.

Since the trivial $\mathfrak{g}$-module $V(0)$ only occurs for $i = 0$, we can plug this into Theorem 1...

Theorem (2)

$$\mathrm{H}^\bullet(\mathfrak{g}[t],\mathbb{C}) \xrightarrow{\sim} \mathrm{H}^\bullet(\mathfrak{g},\mathbb{C}) \text{ as rings.}$$

The isomorphism is the restriction map induced by $ev_0 : \mathfrak{g}[t] \to \mathfrak{g}$.

Feigin (1980) stated that this theorem could be deduced from the Garland-Lepowsky results.

Ext^1

Given $\pi, \pi' \in \mathcal{P}$, let $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\} = \operatorname{supp} \pi \cup \operatorname{supp} \pi'$.

Set $\pi_i = \pi(\mathfrak{m}_i) \in P^+$, so that $\mathcal{V}(\pi) = \otimes_{i=1}^n \mathrm{ev}_{\mathfrak{m}_i}^* V(\pi_i)$; sim. for $\mathcal{V}(\pi')$.

Set $I = \mathfrak{m}_1 \mathfrak{m}_2 \ldots \mathfrak{m}_n \lhd A$; note $\mathfrak{g} \otimes I$ annihilates $\mathcal{V}(\pi), \mathcal{V}(\pi')$.

So $\mathfrak{g}[A]/\mathfrak{g} \otimes I \cong \mathfrak{g}^{\oplus n}$ (by CRT) acts on $\mathcal{V}(\pi), \mathcal{V}(\pi')$.

Using the LHS spectral sequence for $\mathfrak{g} \otimes I \lhd \mathfrak{g}[A]$, Künneth Formula, Whitehead Lemmas, etc., we get:

Theorem (3)

$\mathrm{Ext}^1_{\mathfrak{g}[A]}(\mathcal{V}(\pi), \mathcal{V}(\pi'))$ *is either 0, or is a direct sum of one or more terms of the form* $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\pi_i), V(\pi_i'))$.

Recovers results of Chari-Greenstein (2005: $A = \mathbb{C}[t]$) and Kodera (2010: general A). Also proved by Neher-Savage (2015) in the context of equivariant map algebras.

Ext^2

Theorem (4)

Assume $\pi_i \neq \pi_i'$ for some $1 \leq i \leq n$. Then

$$\mathrm{Ext}^2_{\mathfrak{g}[A]}(\mathcal{V}(\pi), \mathcal{V}(\pi')) \cong \mathrm{Hom}_{\mathfrak{g}[A]/\mathfrak{g}\otimes I}(\mathcal{V}(\pi), \mathrm{H}^2(\mathfrak{g} \otimes I, \mathbb{C}) \otimes \mathcal{V}(\pi')).$$

We have a similar formula for $\mathrm{Ext}^2_{\mathfrak{g}[A]}(\mathcal{V}(\pi), \mathcal{V}(\pi))$.

The main obstruction to understanding Ext^2 between simples is $\mathrm{H}^2(\mathfrak{g} \otimes I, \mathbb{C})$...

Finite Dimensionality of Ext^2

Make the following Standing Assumptions from now on:

- $A = \mathbb{C}[t]$
- $I = \langle (t-a)(t-b) \rangle \triangleleft A$ for distinct $a, b \in \mathbb{C}$

In particular, $\mathcal{V}(\pi)$ and $\mathcal{V}(\pi')$ are tensor products of two evaluation modules (evaluated at a, b).

Theorem (5)

Under the Standing Assumptions, $\mathrm{H}^2(\mathfrak{g} \otimes I, \mathbb{C})$ *is finite-dimensional, and every* $\mathfrak{g}$*-composition factor is of the form* $V(\lambda), \lambda \in P^+, \lambda \leq 2\theta$*.*

Corollary

Under the Standing Assumptions, $\mathrm{Ext}^2_{\mathfrak{g}[t]}(\mathcal{V}(\pi), \mathcal{V}(\pi'))$ *is finite-dimensional.*

The proof of Theorem 5 involves many explicit calculations. At one point, we invoke a result of Zusmanovich (1994) which entails computing the *first cyclic homology* $\mathrm{HC}_1(A') := \Lambda^2(A')/T(A')$ for the algebra $A' := \mathbb{C} \oplus I$, where

$$T(A') = \mathrm{span}\{fg \wedge h + gh \wedge f + hf \wedge g \mid f, g, h \in A'\}.$$

We proved:

Theorem (6)

Under the Standing Assumptions, $\dim \mathrm{HC}_1(A') = 2$.

$\mathfrak{g} \times \mathfrak{g}$ Structure of $H^2(\mathfrak{g} \otimes I, \mathbb{C})$

By analyzing the LHS s.s. for the ideal $\mathfrak{g} \otimes I^s \triangleleft \mathfrak{g} \otimes I$ $(s > 1)$, and passing to the associated graded algebra for the quotient, we deduce:

Proposition

Let $\lambda, \mu \in P^+, \lambda \neq 0$. Then, under the Standing Assumptions,

$$\mathrm{Hom}_{\mathfrak{g}\times\mathfrak{g}}(V(\lambda) \boxtimes V(\mu), H^2(\mathfrak{g} \otimes I, \mathbb{C})) \cong \mathrm{Hom}_{\mathfrak{g}}(V(\lambda), H^2(\mathfrak{g}[t]_+, ev_{b-a}^* V(\mu)^*))$$

Corollary

Let $0 \neq \lambda \in P^+$. Then

$$[\, H^2(\mathfrak{g} \otimes I, \mathbb{C})) : V(\lambda) \boxtimes \mathbb{C}\,]_{\mathfrak{g}\times\mathfrak{g}} = m > 0 \iff \lambda = \lambda_w^*,\ w \in W_a^1,\ \ell(w) = 2,$$

in which case $m = 1$; and similarly for $\mathbb{C} \boxtimes V(\lambda)$.

Also: $[\, H^2(\mathfrak{g} \otimes I, \mathbb{C})) : \mathbb{C} \boxtimes \mathbb{C}\,] = 1$, and $[\, H^2(\mathfrak{g} \otimes I, \mathbb{C})) : \mathfrak{g}^* \boxtimes \mathfrak{g}^*\,] > 0$.

THANK YOU!